Rational degree is polynomially related to degree

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Degree

$f\colon \{0,1\}^n \to \{0,1\}$

Fact. For every $g\colon \{0,1\}^n \to \mathbb{R}$, there exists a unique multilinear real polynomial p that represents g, i.e., $p(x) = g(x)$ for all $x \in \{0,1\}^n$

Definition. The degree of f is defined to be the degree of the unique multilinear real polynomial that represents it. Denoted $\deg(f)$

Example. $n = 2, f(x_1, x_2) = x_1 \oplus x_2$. Represented by

$$p = -2x_1x_2 + x_1 + x_2$$

Rational degree

$f\colon \{0,1\}^n \to \{0,1\}$

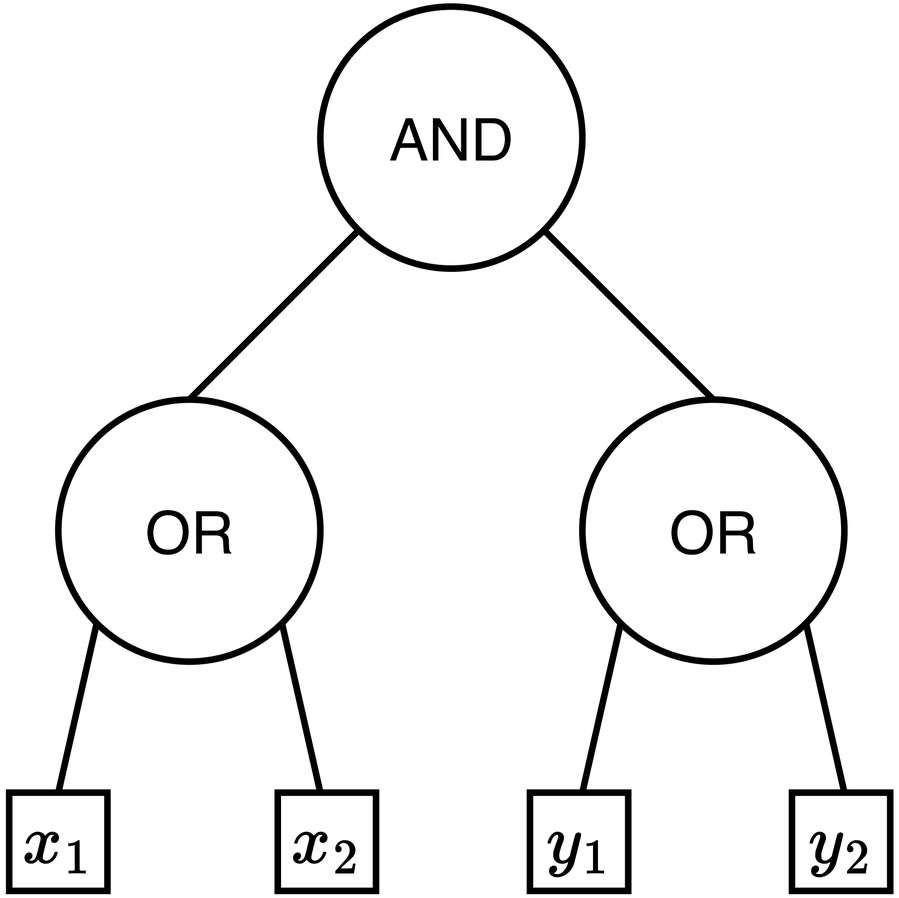
Definition. The rational degree of f is defined as the minimum value of $\max(\deg(p), \deg(q))$, where p, q are multilinear real polynomials such that p/q represents f, i.e., $p(x)/q(x) = f(x)$ for all $x \in \{0,1\}^n$. Denoted $\mathrm{rdeg}(f)$

Observation. $\mathrm{rdeg}(f) \leq \deg(f)$

Example. $n = 2, f(x_1, x_2) = x_1 \oplus x_2$. Can be represented by

$$\frac{p}{q} = \frac{x_1 - x_2}{1 - 2x_2} \quad \text{or} \quad \frac{p}{q} = \frac{x_2 - x_1}{1 - 2x_1}$$

Rational degree can be quadratically smaller than degree



For n square, $f = \mathrm{AND}_{\sqrt{n}} \circ \mathrm{OR}_{\sqrt{n}}$

$$\deg(f) = n \quad \text{and} \quad \mathrm{rdeg}(f) \leq \sqrt{n}$$

Example. $n = 4$

$$\frac{p}{q} = \frac{(x_1 + x_2)(y_1 + y_2)}{(x_1 + x_2)(y_1 + y_2) + [(1 - x_1)(1 - x_2) + (1 - y_1)(1 - y_2)]}$$

Interpretations of rdeg and deg

rdeg: equals exact postselected quantum query complexity $\mathrm{PostQ_E}$

[Mahadev and de Wolf ’15]

[Aaronson ’05]

deg: polynomially related to almost all other complexity measures

e.g., block sensitivity bs and decision tree complexity D

[Nisan and Szegedy ’94]

A question of Fortnow, Nisan, and Szegedy (1994)

For all f: $\{0,1\}^n \to \{0,1\}$, $\deg(f) \leq \mathrm{poly}(\mathrm{rdeg}(f))$?

(Quantumness + postselection isn't much more powerful than deterministic?)

Prior results:

1. Randomness + postselection isn't much more powerful than deterministic: $\mathrm{D}(f) \leq \mathrm{PostR}_{\mathrm{E}}(f)^2$ [Nisan '89, Cade '20]

2. Yes for f symmetric or monotone [Iyer, Jain, Kothari, Kovacs-Deak, Kumar, Schaeffer, Wang, Whitmeyer '23]

We resolve the question affirmatively

Theorem

For all $f\colon \{0,1\}^n \to \{0,1\}$, $\deg(f) \leq \widetilde{O}(\mathrm{rdeg}(f)^3)$

[KKWY ’26]

A corollary relating three degrees

$\deg(f)$: degree of polynomial representing f over $\mathbb{R}$

query complexity and analysis of Boolean functions

$$\deg(f) \leq \widetilde{O}(\deg_{\mathbb{F}_2}(f) \cdot \deg_{\pm}(f)^2)$$

$\deg_{\mathbb{F}_2}(f)$: degree of polynomial representing f over $\mathbb{F}_2$

coding theory and cryptography

$\deg_{\mathbb{F}_2}(f) \leq \deg(f)$

$\deg_{\pm}(f)$: minimum degree of polynomial representing f in sign

learning theory and complexity theory

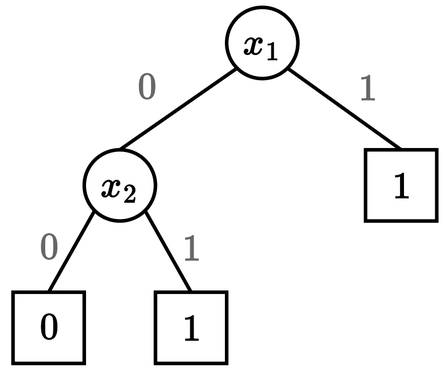
$\deg_{\pm}(f) \leq \deg(f)$

Proof of $\deg(f) \leq 3\,\mathrm{rdeg}(f)^4$

Key takeaway: best-case complexity measure

Decision tree complexity and nondeterministic degree

$f\colon \{0,1\}^n \to \{0,1\}$



$\mathrm{D}(f)$: minimum depth of a decision tree computing f

Fact. $\deg(f) \le \mathrm{D}(f)$. **Proof by example.** $x_1 + (1 - x_1)\, x_2$

$\mathrm{ndeg}(f)$: minimum degree of a nondeterministic polynomial p for f, i.e., p multilinear real and $p(x) = 0 \iff f(x) = 0$ for all $x \in \{0,1\}^n$

Fact. $\mathrm{ndeg}(f), \mathrm{ndeg}(\neg f) \le \mathrm{rdeg}(f)$. **Proof.** $f = p/q \implies \mathrm{ndeg}(f) \le \deg(p), \mathrm{ndeg}(\neg f) \le \deg(p - q)$

Theorem. For all $f\colon \{0,1\}^n \to \{0,1\}$, $\mathrm{D}(f) \le 3\, \mathrm{ndeg}(f)^2\, \mathrm{ndeg}(\neg f)^2$

A decision tree

$f\colon \{0,1\}^n \to \{0,1\}$; let p be a nondeterministic polynomial for f of degree $\mathrm{ndeg}(f)$

Repeat

1. query variables in p such that its degree decreases by at least 1

2. set variables in p to their queried values, call resulting polynomial p

3. if p is a constant c, output 0 if $c = 0$, output 1 if $c \neq 0$

Observation

If can query no more than H variables in step 1, then $\mathrm{D}(f) \leq H \cdot \mathrm{ndeg}(f)$

Keep querying disjoint maxonomials of p

Definition. A maxonomial of p is a monomial of p with maximum degree. Two maxonomials are said to be disjoint when they do not share any variable

Example. $p = 2X_1X_2X_3 + 3X_3X_4X_5 - X_5X_6X_7 - X_1X_2 - X_3$

A maximal set of disjoint maxonomials: $\{X_1X_2X_3, X_5X_6X_7\}$

Observations. (i) degree of p must decrease by at least 1, (ii) number of queries is at most $\deg(p) \times$ (number of disjoint maxonomials) [Nisan and Smolensky '98]

Bounding number of disjoint maxonomials

Definition. The block sensitivity of $f\colon \{0,1\}^n \to \{0,1\}$ at $x \in \{0,1\}^n$ is the largest k such that there exist k pairwise disjoint blocks of variables (sensitive blocks) where flipping all the bits of x in any one block changes the value of f

Denoted $\mathrm{bs}_x(f)$

[Nisan '89]

Example. $n = 3$ and $f(x) = x_1 \vee x_2 \vee x_3$

1. $\mathrm{bs}_{000}(f) = 3\colon B_1 = \{1\}, B_2 = \{2\}, B_3 = \{3\}$
2. $\mathrm{bs}_{111}(f) = 1\colon B_1 = \{1,2,3\}$

Bounding number of disjoint maxonomials

Recall $f\colon \{0,1\}^n \to \{0,1\}$; p a nondeterministic polynomial for f

Lemma. Suppose f is nonconstant. Then, the number of disjoint maxonomials of p is at most $\min_{x \in f^{-1}(0)} \mathrm{bs}_x(f)$ (a best-case complexity measure)

Proof by example. Consider the $f\colon \{0,1\}^7 \to \{0,1\}$ nondeterministically represented by $p = 2X_1X_2X_3 + 3X_3X_4X_5 - X_5X_6X_7 - X_1X_2 - X_3$

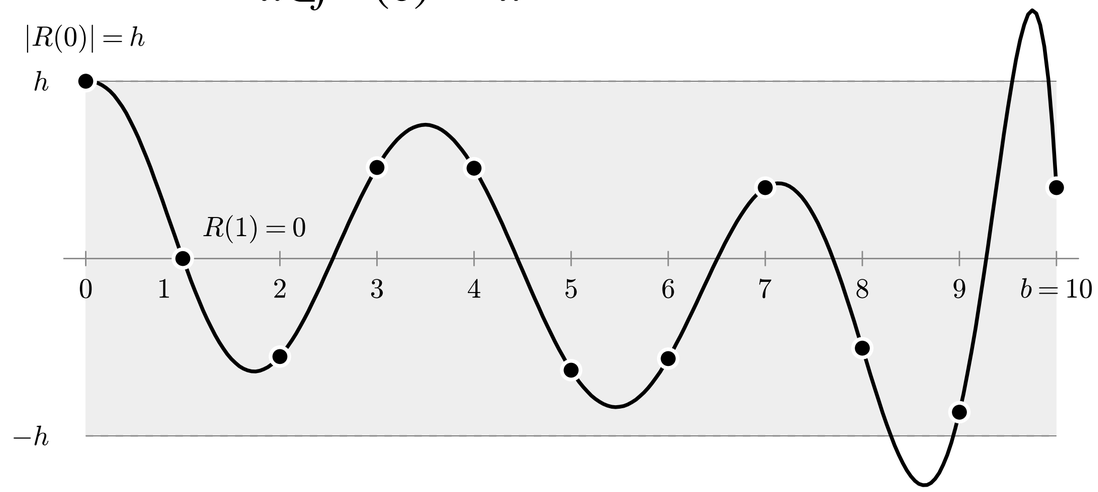
Fix arbitrary $y \in f^{-1}(0)$. Consider maxonomial $X_1X_2X_3$: set remaining variables in p according to y. Resulting polynomial nonzero $\Longrightarrow$ $X_1X_2X_3$ contains a sensitive block for y. Every maxonomial contains a sensitive block for y

Bounding minimum block sensitivity

Lemma. Let $f\colon \{0,1\}^n \to \{0,1\}$. Suppose f is nonconstant, then

$$\min_{x \in f^{-1}(0)} \mathrm{bs}_x(f) \leq 3\,\mathrm{ndeg}(\neg f)^2$$

Proof. Take nondeterministic polynomial p for $\neg f$, so $p(x) \neq 0 \iff f(x) = 0$.
Let $h = \max_{x \in f^{-1}(0)} |p(x)|$ and $z \in f^{-1}(0)$ be such that $|p(z)| = h$. Write $b = \mathrm{bs}_z(f)$.
Then $b \geq \min_{x \in f^{-1}(0)} \mathrm{bs}_x(f)$



Symmetrizing around z gives real univariate polynomial R with $\deg(R) \leq \deg(p)$ like in the figure

Standard argument gives $b \leq 3\deg(R)^2 \leq 3\deg(p)^2$

Putting things together

$f\colon \{0,1\}^n \to \{0,1\}$; let p be a nondeterministic polynomial for f of degree $\mathrm{ndeg}(f)$

Repeat

1. query variables in p such that its degree decreases by at least 1

2. set variables in p to their queried values, call resulting polynomial p

3. if p is a constant c, output 0 if $c = 0$, output 1 if $c \neq 0$

Observation: If can query no more than H variables in step 1, then $\mathrm{D}(f) \leq H \cdot \mathrm{ndeg}(f)$

$$
\begin{aligned}
H &\leq \deg(p) \cdot (\text{no. disjoint maxonomials}) \\
&\leq \deg(p) \cdot \min_{x \in f^{-1}(0)} \mathrm{bs}_x(f) \\
&\leq \deg(p) \cdot 3\,\mathrm{ndeg}(\neg f)^2
\end{aligned}
$$

so $\mathrm{D}(f) \leq 3\,\mathrm{ndeg}(f)^2 \cdot \mathrm{ndeg}(\neg f)^2$

Two open questions

Question. Is $\deg(f) \leq \widetilde{O}(\mathrm{rdeg}(f)^3)$ tight?

Generalize beyond the Boolean cube

Let $X \subseteq \mathbb{R}^n$. For $f\colon X \to \{0,1\}$, define $\deg(f), \mathrm{rdeg}(f)$ analogously

Question. For given X, does $\deg(f) \leq \mathrm{poly}(\mathrm{rdeg}(f))$ hold?

Our result shows answer is yes for $X = \{0,1\}^n$

Our paper also shows answer is no for some subset $X \subseteq \{0,1\}^n$

Questions of this flavor are studied in math: effective Nullstellensatz
[Brownawell ’87, Kollár ’88, Alon ’99, Jelonek ’05]

Thank you for your attention



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Theorem. For all $f\colon \{0,1\}^n \to \{0,1\}$, $\deg(f) \leq \widetilde{O}(\mathrm{rdeg}(f)^3)$

Improving the bound

Improving the bound (sketch)

Definition. A hitting set of a polynomial p is a set of variables that intersect every maxonomial of p. Smallest size of such set denoted $\mathrm{H}(p)$

Previously: for p nondeterministic polynomial for $f\colon \{0,1\}^n \to \{0,1\}$

1. $\mathrm{D}(f) \leq \deg(p) \cdot \mathrm{H}(p)$
2. $\mathrm{H}(p) \leq \deg(p) \cdot \min_{x \in f^{-1}(0)} \mathrm{bs}_x(f)$
3. $\min_{x \in f^{-1}(0)} \mathrm{bs}_x(f) \leq O(\mathrm{ndeg}(\neg f)^2)$

I. $\mathrm{D}(f) \leq O(\deg(p) \cdot \mathrm{RC}^{\downarrow}_{0,\min}(f) \cdot \log n)$

inspired by [Aaronson '08]

II. Skip!

III. $\min_{x \in f^{-1}(0)} \mathrm{fbs}_x(f) \leq O(\mathrm{ndeg}(\neg f)^2)$

inspired by [Anshu, Ben-David, Kundu '21]

Improving the bound (sketch)

Recall $\mathrm{D}(f) \leq O(\deg(p) \cdot \mathrm{RC}^{\downarrow}_{0,\min}(f) \cdot \log n)$ and $\min_{x \in f^{-1}(0)} \mathrm{fbs}_x(f) \leq O(\mathrm{ndeg}(\neg f)^2)$

where p is a nondeterministic polynomial for f

But $\mathrm{RC}^{\downarrow}_{0,\min}(f) = \Theta(\min_{x \in f^{-1}(0)} \mathrm{fbs}_x(f))$ by LP duality [Tal '13, Gilmer, Saks, Srinivasan '16]

So $\mathrm{D}(f) \leq O(\mathrm{ndeg}(f) \cdot \mathrm{ndeg}(\neg f)^2 \cdot \log n)$, so $\mathrm{D}(f) \leq O(\mathrm{rdeg}(f)^3 \cdot \log n)$

So $\deg(f) \leq O(\mathrm{rdeg}(f)^3 \cdot \log \mathrm{rdeg}(f))$