Lecture 7

Tsirelson's inequality.

Assume for simplicity that Alice and Bob use a 2-qubit state $|\psi\rangle$. Really, this is just to make the notation less cluttered, there is essentially no loss in content.

Suppose that Alice applies U_x given question $x \in \{0,1\}$ and Bob applies V_y given question $y \in \{0,1\}$.

Consider

$$\begin{aligned}
&|\langle 00| U_x \otimes V_y |\psi\rangle|^2 + |\langle 11| U_x \otimes V_y |\psi\rangle|^2 \\
&= \langle\psi| U_x^\dagger \otimes V_y^\dagger |00\rangle\langle 00| U_x \otimes V_y |\psi\rangle + (\cdot) \qquad (z \in \mathbb{C} \implies |z|^2 = z^\dagger z) \\
&= \langle\psi| U_x^\dagger |0\rangle\langle 0| U_x \otimes V_y^\dagger |0\rangle\langle 0| V_y |\psi\rangle + (\cdot) \\
&= \langle\psi| P_x^0 \otimes Q_y^0 |\psi\rangle + \langle\psi| P_x^1 \otimes Q_y^1 |\psi\rangle
\end{aligned}$$

where

$$P_x^b := U_x^\dagger |b\rangle\langle b| U_x \quad \text{and} \quad Q_y^b := V_y^\dagger |b\rangle\langle b| V_y \tag{1}$$

Similarly,

$$|\langle 01| U_x \otimes V_y |\psi\rangle|^2 + |\langle 10| U_x \otimes V_y |\psi\rangle|^2 = \langle\psi| P_x^0 \otimes Q_y^1 |\psi\rangle + \langle\psi| P_x^1 \otimes Q_y^0 |\psi\rangle \tag{2}$$

Now, observe that

$$\begin{aligned}
&\langle\psi| P_x^0 \otimes Q_y^0 |\psi\rangle + \langle\psi| P_x^1 \otimes Q_y^1 |\psi\rangle - \langle\psi| P_x^0 \otimes Q_y^1 |\psi\rangle + \langle\psi| P_x^1 \otimes Q_y^0 |\psi\rangle \\
&= \langle\psi| \left(P_x^0 \otimes Q_y^0 + P_x^1 \otimes Q_y^1 - P_x^0 \otimes Q_y^1 - P_x^1 \otimes Q_y^0\right) |\psi\rangle \\
&= \langle\psi| \left(P_x^0 - P_x^1\right) \otimes \left(Q_y^0 - Q_y^1\right) |\psi\rangle \\
&= \langle\psi| \left(A_x \otimes B_y\right) |\psi\rangle,
\end{aligned}$$

where

$$A_x := P_x^0 - P_x^1 \quad \text{and} \quad B_y := Q_y^0 - Q_y^1 \tag{3}$$

Recall that when $(x,y) \in \{(0,0),(0,1),(1,0)\}$, the winning answers are $(a,b) \in \{(0,0),(1,1)\}$; when $(x,y) = (1,1)$, the winning answers are $(a,b) \in \{(0,1),(1,0)\}$. Therefore, the winning probability minus the losing probability is

$$\frac{1}{4} \langle\psi| \Big(A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1\Big) |\psi\rangle. \tag{4}$$

Observe that A_x, B_y are Hermitian and satisfy

$$A_x^2 = I_2 \quad \text{and} \quad B_y^2 = I_2. \tag{5}$$

Comment: If Alice and Bob use a d^2-dimensional quantum state, then above will simply be I_d, and the rest of the derivation goes through since it only uses the fact I_d is the multiplicative matrix identity.

Lemma 1. *Write $C := A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$, then*

$$C^2 = 4I_4 - [A_0, A_1] \otimes [B_0, B_1], \tag{6}$$

where $[\cdot,\cdot]$ denotes the commutator.

Definition 1. Let $A \in \mathbb{C}^{m\times n}$ be a complex matrix. The spectral norm of A, denoted $\|A\|$ is defined to be $\max_{0\neq u\in\mathbb{C}^n} \|Au\|/\|u\|$.

Comment: We're not using all these later but this is a good place to record these facts, which are very useful throughout QI/QC.

Fact 1. The spectral norm satisfies the following properties. Let A, B be a complex matrix. Let $|u\rangle, |v\rangle$ be column vectors. Let $\alpha \in \mathbb{C}$. Then, assuming all dimensions are compatible, we have

1. Spectral norm is a norm: (i) $\|A|u\rangle\| \geq 0$ with equality if and only if $|u\rangle = 0$; (ii) $\|\alpha A\| = |\alpha|\|A\|$; (iii) $\|A+B\| \leq \|A\| + \|B\|$.
2. $\|A \otimes B\| = \|A\|\|B\|$ Comment: the $\leq$ direction was very useful in a recent research paper!
3. $|\langle u| A |v\rangle| \leq \|A\|\|u\|\|v\|$
4. Submultiplicativity: $\|AB\| \leq \|A\|\|B\|$.
5. If $m = n$ and A is Hermitian, then $\|A^2\| = \|A\|^2$.

Observe that C is Hermitian, so

$$|\langle\psi| C |\psi\rangle| \le \|C\| = \sqrt{\|C^2\|} \le \sqrt{4+4} = \sqrt{8}. \tag{7}$$

Write W for the winning probability and L for the losing probability. Then, $W - L \le |W - L| \le \sqrt{8}/4 = \sqrt{2}/2$. Also $W + L = 1$. Therefore, adding gives $2W \le 1 + \sqrt{2}/2$ and thus

$$W \le \frac{2+\sqrt{2}}{4} = \cos^2(\pi/8). \tag{8}$$