CPSC 436Q: Homework 1

Due on Gradescope by 23:59pm on October 1, 2025

Rules

1. Please try to solve the problems yourself first. If you get stuck, you may *consult* any *non-GenAI* resources (books, Wikipedia, lecture notes, peers, office hours, etc.) for solutions. Provided you *acknowledge* these resources, no marks will be deducted. However, you *must* write up your own solution *independently*, using your own words. Answers suspected of being from GenAI will receive zero credit unless you can demonstrate understanding upon appeal.
2. Please write legibly, work that is illegible will be marked as incorrect. Latex is strongly recommended for legibility. (I also recommend using `https://www.overleaf.com/` if you're new to Latex.)
3. All answers should be justified to receive any credit.
4. The total number of points for non-bonus questions is $T = 28$. Credit policy for bonus questions: suppose you receive x points for bonus questions and y points for non-bonus questions, then the total number of points you receive for this homework is $\min(x + y, T)$. Points for bonus questions are generally harder to earn.
5. If you spot any mistakes, please email me at `wdaochen@cs.ubc.ca`. Any corrections will be announced on Piazza.

Homework

1. **Prerequisites.**
   (a) **Symmetric matrices.**
      i. **(2 points)** Show that all eigenvalues of a symmetric real matrix are real.
      ii. **(2 points)** Does the above still hold if the word "symmetric" is dropped. (If true, show it. If false, give a counterexample.)

   (b) **Eigenvalues and eigenvectors.** Let $\theta \in \mathbb{R}$ and $A \in \mathbb{C}^{2\times 2}$ be defined by
   $$A \coloneqq \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{1}$$
      i. **(4 points)** Calculate the eigenvalues and eigenvectors of A. Therefore, write A in the form $A = UDU^\dagger$, where $U \in \mathbb{C}^{2\times 2}$ is unitary and $D \in \mathbb{C}^{2\times 2}$ is diagonal.
      ii. **(2 points)** For $k \in \mathbb{N}$, show that $A^k = UD^kU^\dagger$ and use the expression on the right-hand side to calculate A^k, simplifying your answer as much as possible.

   (c) **Probability.** You have 1000 distinct songs in a playlist. You play 1000 songs, where at each play you sample one of the 1000 songs *uniformly at random with replacement.*
      i. **(4 points)** Which probability is bigger:
         A. each song comes up exactly once,
         B. there exists a song that comes up at least twice.

         Hint: As stated in the rules, you need to justify your answer to receive any credit. You can't just answer A/B.
      ii. **(2 points)** For a fixed song, what is the expected number of times it comes up?

2. **Deterministic complexity of the NAND tree.** We sketched in class that the randomized complexity of the depth-h NAND tree on $n \coloneqq 2^h$ input bits is $O\big(((1+\sqrt{33})/4)^h\big) = O(n^{0.754})$.

   We now consider how well deterministic algorithms perform for this problem. Suppose the n input bits to the NAND tree are $x_1, \ldots, x_n$. These bits are located in fixed order from left to right, that is, x_1 is at the left-most leaf, x_2 is at the leaf immediately to the right of x_1, and so on, until x_n is at the right-most leaf.

(4 points) Let $\{i_1, \ldots, i_{n-1}\}$ be a subset of $\{1, \ldots, n\}$ of size $n-1$. Suppose a deterministic algorithm examines bits $x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}}$. Show that there always exists some assignment of values to those bits (e.g., $x_{i_1} = 0$, $x_{i_2} = 1$, ..., $x_{i_{n-1}} = 0$) such that the output value of the NAND tree, *given* this assignment, still *changes* depending on whether the one remaining unexamined bit is 0 or 1. Hint: it may help to consider the case when $n = 2$ first.

This argument shows that the deterministic complexity of the NAND tree on n input bits is at least n because, in the worst case, a deterministic algorithm has to examine all n bits to know for sure what the output of the NAND tree is.[1]

3. **Kronecker product.** Hint: it's easier to do the following problems if you use Dirac notation as much as possible.

   (a) **(2 points)** Define $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ by

   $$|\psi\rangle := \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle |i\rangle . \tag{2}$$

   Let $\mathbb{1}_d \in \mathbb{C}^{d \times d}$ denote the identity matrix. Show that for any $A \in \mathbb{C}^{d \times d}$, we have

   $$A \otimes \mathbb{1}_d |\psi\rangle = \mathbb{1}_d \otimes A^\top |\psi\rangle , \tag{3}$$

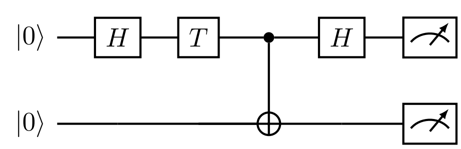
   where $^\top$ denotes the transpose.

   (b) **(2 points)** Let $|u_1\rangle, \ldots, |u_d\rangle \in \mathbb{C}^d$ be an arbitrary orthonormal basis. Show that

   $$|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |u_i\rangle |u_i^*\rangle , \tag{4}$$

   where $|u_i^*\rangle$ denotes the (entry-wise) complex conjugate of $|u_i\rangle$.

4. **Quantum circuit.** Consider the following quantum circuit.



   (a) **(2 points)** Calculate the state of the circuit just before the measurement.

   (b) **(2 points)** The measurement symbols denote measurement in the computational basis. Write down the four probabilities corresponding to obtaining each of the four measurement outcomes $00, 01, 10, 11$.

5. **Bonus question: double-slit experiment.** The laws of quantum mechanics stipulate that the amplitude for a particle of momentum $p \in \mathbb{R}$ to move from position $\vec{r}_1 \in \mathbb{R}^3$ to $\vec{r}_2 \in \mathbb{R}^3$ is given by

   $$\langle \vec{r}_1 \mid \vec{r}_2 \rangle := \exp(ipr_{12}/\hbar)/r_{12}, \tag{5}$$

   where $\hbar$ is the reduced Planck's constant (look it up!) and $r_{12} := \|\vec{r}_1 - \vec{r}_2\|$ is the Euclidean distance between $\vec{r}_1$ and $\vec{r}_2$ in SI units, i.e., meters in this case.

   In the double-slit experiment, let $\vec{s} \in \mathbb{R}^3$ denote the position of the particle source, let $\vec{L} \in \mathbb{R}^3$ and $\vec{R} \in \mathbb{R}^3$ denote the positions of the left and right slits, let $\vec{x} \in \mathbb{R}^3$ denote a position on the screen. Then, the laws of quantum mechanics also stipulate that the probability of finding the particle at position $\vec{x}$ is given by

   $$P(\vec{x}) := \left| \langle \vec{x} | \vec{L} \rangle \cdot \langle \vec{L} | \vec{s} \rangle + \langle \vec{x} | \vec{R} \rangle \cdot \langle \vec{R} | \vec{s} \rangle \right|^2 \tag{6}$$

   (The two terms in the sum correspond to the two ways for the particle to reach $\vec{x}$ from the source, one way through the left slit, another through the right slit. You'll see that these two terms can *subtract*!)

   Suppose now that the source is equidistant from the two slits and its distance to each is 1 meter. Suppose that the distance between the two slits is $2d$ meters. Suppose that the perpendicular distance from the slits to the screen is l meters. Suppose x is the distance from the center of the screen to $\vec{x}$.

   (4 points) Use eq. (5) to give a simplified real expression for $P(\vec{x})$ in terms of d, l, x, p (and the constant $\hbar$). Hint: draw a picture!

   (You can now play around with your simplified expression to see how different values of d, l, x, p lead to different interference patterns. No points here but it's fun! In particular, you can use the expression to understand why "no interference" is observed for macroscopic objects like bullets – the truth is that there are interference patterns but the interference fringes are too close together to be detectable; think about the p value for a bullet versus the p value for a small particle like an electron – you may need to do some Googling.)

[1] More precisely, it shows that the *deterministic query complexity* of the NAND tree is n.