CPSC 436Q: Homework 2

Due on Gradescope by 11:59pm on November 5, 2025

Rules

1. Please try to solve the problems yourself first. If you get stuck, you may *consult* any *non-GenAI* resources (books, Wikipedia, lecture notes, peers, office hours, etc.) for solutions. Provided you *acknowledge* these resources, no marks will be deducted. However, you *must* write up your own solution *independently*, using your own words. Answers suspected of being from GenAI will receive zero credit unless you can demonstrate understanding upon appeal.
2. Please write legibly, work that is illegible will be marked as incorrect. Latex is strongly recommended for legibility. (I also recommend using `https://www.overleaf.com/` if you're new to Latex.)
3. All answers should be justified to receive any credit.
4. The total number of points for non-bonus questions is $T = 28$. Credit policy for bonus questions: suppose you receive x points for bonus questions and y points for non-bonus questions, then the total number of points you receive for this homework is $\min(x + y, T)$. Points for bonus questions are generally harder to earn.
5. If you spot any mistakes, please email me at `wdaochen@cs.ubc.ca`. Any corrections will be announced on Piazza.

Homework

1. **Unitarity of the quantum Fourier transform.** For a positive integer d, the d-dimensional quantum Fourier transform is the $d \times d$ matrix defined by

$$\mathrm{QFT}_d \left|i\right\rangle = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} \omega_d^{i \cdot j} \left|j\right\rangle, \quad \text{for all } i \in \{0, 1 \ldots, d-1\}, \tag{1}$$

   where $\omega_d \coloneqq \exp(2\pi i/d)$. Note that it is implicit here that $\left|i\right\rangle$ denotes the $(i + 1)$th standard basis vector of $\mathbb{C}^d$.

   (4 points) Show that QFT_d is unitary for all d.

2. **Entangled or not?** Recall that a two-qubit state is said to be *not entangled* if and only if it can be written as a tensor product of two one-qubit states. That is, $\left|\psi\right\rangle \in \mathbb{C}^4$ is not entangled if and only if $\left|\psi\right\rangle = \left|\psi_1\right\rangle \otimes \left|\psi_2\right\rangle$ where $\left|\psi_i\right\rangle \in \mathbb{C}^2$.

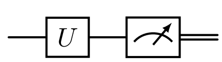
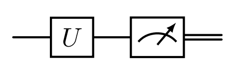
   (2 points) Show that the EPR pair $\frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle)$ is entangled.

   Are the following states entangled? Justify your answers.

$$\textbf{(2 points)} \quad \frac{1}{\sqrt{3}}(\left|00\right\rangle + \left|11\right\rangle + \left|01\right\rangle), \tag{2}$$

$$\textbf{(2 points)} \quad \frac{\sqrt{2}}{3}\left|00\right\rangle - \frac{1}{3}\left|01\right\rangle + \frac{2}{3}\left|10\right\rangle - \frac{\sqrt{2}}{3}\left|11\right\rangle. \tag{3}$$

3. **Optimal measurement.** Consider the following two single-qubit circuits:

   (i) $\left|0\right\rangle$  and (ii) $\left|+\right\rangle$  (4)

   Let p denote the probability that the measurement in circuit (i) returns 0.

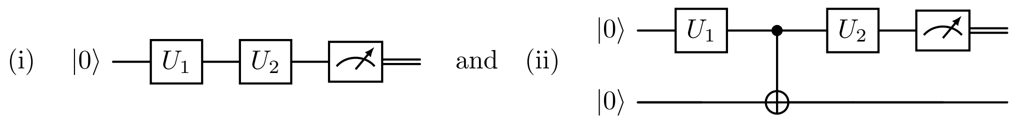
   Let q denote the probability that the measurement in circuit (ii) returns 1.

   (6 points) Construct a 2×2 unitary matrix U such that the value of

$$\min(p, q) \tag{5}$$

   is as large as possible. The grade you get scales with the value you get. Full marks will be given if you get the maximum possible value (you do not need to justify why it's the maximum). Hint: think geometrically, like what we did in our second analysis of the CHSH game's quantum strategy.

4. **Partial measurement.** Consider the following two quantum circuits, where circuit (i) acts on one qubit and circuit (ii) acts on two qubits.



(6)

(6 points) Is it true that the measurement distribution of circuit (i) is the same as the (partial) measurement distribution of circuit (ii), irrespective of the choice of the 2×2 unitary matrices U_1 and U_2? If yes, give a proof. If not, give and explain a counterexample.

5. **Quantum oracle instantiation.** Suppose $C\colon \{0,1\}^3 \to \{0,1\}$ is defined by

$$C(x) = \text{NAND}(\text{NAND}(x_1, x_2), x_3), \tag{7}$$

where $\text{NAND}(a, b) := \text{NOT}(\text{AND}(a, b))$ as per usual.

(6 points) Draw a quantum circuit that implements the unitary $Q\colon |x\rangle |b\rangle \mapsto |x\rangle |b \oplus C(x)\rangle$ for all $x \in \{0,1\}^3$ and $b \in \{0,1\}$. You should follow these rules:

(a) You are only allowed to use X, CNOT, and Toffoli gates.

(b) Your circuit should use six qubits, including two ancilla qubits that both start and end in the state $|0\rangle$.

(c) Justify why your circuit works as desired.

6. **Bonus question: circuit compilation.** The S gate implements the unitary

$$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}. \tag{8}$$

(4 points) Prove or disprove the following statement.

For all unitaries $U \in \mathbb{C}^{2\times 2}$ and for all $\epsilon > 0$, there exists a quantum circuit on 1 qubit defined by a finite sequence of H (Hadamard) and S gates such that the unitary $V \in \mathbb{C}^{2\times 2}$ implemented by the quantum circuit satisfies

$$\|V - U\|_F \leq \epsilon, \tag{9}$$

where $\|\cdot\|_F$ denotes the Frobenius norm, i.e.,

$$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\|_F := \sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}. \tag{10}$$

[To receive any credit for this problem, you must prove/disprove from first principles. You may not invoke well-known theorems.]

Remark 1. *In the jargon, this question is asking whether the gate set $\{H, S\}$ is universal for single-qubit unitaries.*