Lecture 10

More comments about CHSH game in the context of physics. Search for loop-hole free Bell tests for more!

Definition 11 (Partial measurement)**.** Let $m \leq n$ be positive integers. A partial measurement of the first m qubits of an n-qubit state $|\psi\rangle := \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle$ is a process that

1. Returns $y \in \{0,1\}^m$ with probability

$$p_y := \|(\langle y| \otimes \mathbb{1}^{\otimes(n-m)}) |\psi\rangle \|^2 = \sum_{x|x[1:m]=y} |\alpha_x|^2 = \sum_{z\in\{0,1\}^{n-m}} |\alpha_{yz}|^2 \tag{54}$$

   (Note $\mathbb{1}^{\otimes(n-m)} = \mathbb{1}_{2^{n-m}}$.)

2. Given $y \in \{0,1\}^m$ is returned, the state of the remaining $n-m$ qubits becomes ("collapses to")

$$|\psi_y\rangle := \frac{(\langle y| \otimes \mathbb{1}_{2^{n-m}}) |\psi\rangle}{\|(\langle y| \otimes \mathbb{1}_{2^{n-m}}) |\psi\rangle \|} = \frac{1}{\sqrt{p_y}} \sum_{z\in\{0,1\}^{n-m}} \alpha_{yz} |z\rangle , \tag{55}$$

   where yz denotes y concatenated with z.

(Natural generalization to a subset S of qubits $x[1{:}\, m] \to x_S$, $\langle y| \otimes \mathbb{1}^{\otimes(n-m)} \to \langle y|_S \otimes \mathbb{1}^{\otimes[n]-S}$.)

Example:

$$|\psi\rangle = \sqrt{1/2}\,|00\rangle + \sqrt{1/4}\,|01\rangle + \sqrt{1/6}\,|10\rangle + \sqrt{1/12}\,|11\rangle . \tag{56}$$

Exercise: 1. check the (non-colon)equalities, 2. suppose you take an n-qubit measurement, and measure the qubits one at a time, then the resulting probability distribution is the same as the probability distribution from measuring them all at once.

Using partial measurements to understand the CHSH quantum strategy more intuitively. Recall: for two real vectors $\vec{u} = |u\rangle$ and $\vec{v} = |v\rangle$ of the same dimension,

$$\langle u|v\rangle = \vec{u} \cdot \vec{v} = \|u\|\|v\| \cos(\varphi), \tag{57}$$

where φ is the angle between $|u\rangle , |v\rangle$ Comment: can be proven using law of cosines, say: $c^2 = a^2+b^2-2ab\cos\gamma$. In particular, if $|u\rangle$ and $|v\rangle$ are unit vectors, we have $|\langle u|v\rangle|^2 = \cos^2(\varphi)$.

Observe that

$$U_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{58}$$

is a rotation matrix by angle θ clockwise.

Then do the cases $(x, y) = (0, 0)$ and $(x, y) = (1, 0)$ again.

1. Case $(x, y) = (0, 0)$.
   (a) Alice measures 0. Analysis

$$(\langle 0| \otimes \mathbb{1}_2)\, \mathbb{1} \otimes U_{\pi/8} |\text{EPR}\rangle = \langle 0| \otimes U_{\pi/8} |\text{EPR}\rangle = \frac{1}{\sqrt{2}} U_{\pi/8} |0\rangle . \tag{59}$$

   So probability is 1/2, and given this outcome, the state (on Bob's side) collapses to

$$U_{\pi/8} |0\rangle . \tag{60}$$

   So the probability of Bob measuring 0 is $\cos^2(\pi/8)$, as can be seen by a 2D picture.

   (b) Alice measures 1. Similar.

2. Case $(x, y) = (0, 1)$. Start with the key observation that

$$|\text{EPR}\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|+\rangle |+\rangle + |-\rangle |-\rangle), \tag{61}$$

   since $\{|+\rangle , |-\rangle\}$ forms an orthonormal basis for $\mathbb{C}^2$ and are real (HW Q3(b)). Comment: or just check explicitly!

(a) Alice measures 0. Analysis

$$((\langle 0| \otimes \mathbb{1}_2) H \otimes U_{\pi/8} |\text{EPR}\rangle = \langle +| \otimes U_{\pi/8} |\text{EPR}\rangle = \frac{1}{\sqrt{2}} U_{\pi/8} |+\rangle \tag{62}$$

So probability is 1/2, and given this outcome, the state (on Bob's side) collapses to

$$U_{\pi/8} |+\rangle . \tag{63}$$

So the probability of Bob measuring 0 is $\cos^2(\pi/8)$ as can be seen by a 2D picture.

(b) Alice measures 1. Similar.