Lecture 22

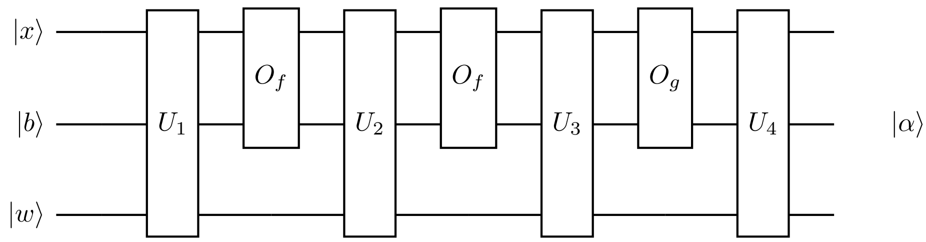
Claim 2. $\| |\psi\rangle - |\gamma\rangle \| \leq 2\sum_{t=1}^{T} \sqrt{Q_{x^*}^t}$

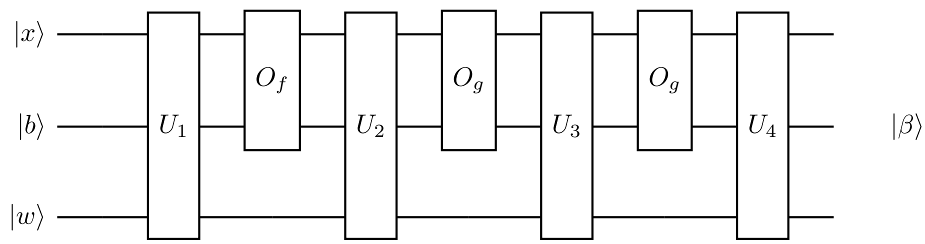
Proof. By direct calculation (did in class), we have

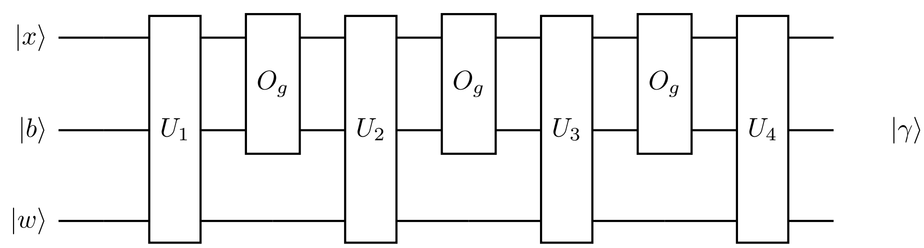
$$\| |\psi\rangle - |\alpha\rangle \|^2 \leq 4Q_{x^*}^3 \tag{129}$$

$$\| |\alpha\rangle - |\beta\rangle \|^2 \leq 4Q_{x^*}^2 \tag{130}$$

$$\| |\beta\rangle - |\gamma\rangle \|^2 \leq 4Q_{x^*}^1 \tag{131}$$







□

Therefore,

$$\| |\psi\rangle - |\gamma\rangle \| \leq 2\sum_{t=1}^{T} \sqrt{Q_{x^*}^t} \leq 2\sqrt{T}\sqrt{\sum_{t=1}^{T} Q_{x^*}^t} \leq 2\frac{T}{\sqrt{N}}, \tag{132}$$

which is small unless $T \geq \Omega(\sqrt{N})$.

Quantum error correction Comment: Discuss classical and quantum error correction briefly, then classical bit flip code and Hamming code.

No cloning makes QEC non-trivial. No unitary that maps $|\psi\rangle|0\rangle|0\rangle$ to $|\psi\rangle|\psi\rangle|\psi\rangle$ *for all* $|\psi\rangle$. Note it is possible if the possibilities for $|\psi\rangle$ are restricted, e.g., $|\psi\rangle \in \{|0\rangle, |1\rangle\}$.

Definition 21. A d-dimensional observable O is a $d \times d$ Hermitian matrix, i.e., $O^\dagger = O$. An n-qubit observable is a 2^n-dimensional observable

Definition 22. Let O be an n-qubit observable and $|\psi\rangle$ and n-qubit quantum state. Suppose $O|\psi\rangle = \lambda|\psi\rangle$ for some real λ, then measuring O on $|\psi\rangle$ is a process that returns λ with certainty and the state remains $|\psi\rangle$.

Example 3. *Measure* Z *on* $|0\rangle$*. Measure* X *on* $|+\rangle$

Remark 8. (Non-examinable.) Can also define measuring O on non-eigenstates $|\psi\rangle$. Write $O = \sum_k \lambda_k \Pi_k$, where λ_k are the distinct eigenvalues of k and Π_k are the orthogonal projectors onto the corresponding eigenspace. Then measuring $|\psi\rangle$ returns λ_k with probability $\langle\psi| \Pi_k |\psi\rangle = \|\Pi_k |\psi\rangle\|^2$ and the state collapses to $\Pi_k |\psi\rangle / \|\Pi_k |\psi\rangle\|$. Consistent with our usual measurement notion: measure $Z \otimes I \otimes \cdots \otimes I$, then ..., then $I \otimes \cdots \otimes I \otimes Z$. (Technically, under relabelling $0 \mapsto 1$ and $1 \mapsto -1$.) Also consistent with definition above, which covers the special case when $|\psi\rangle$ is an eigenstate.

For this course, if O is measured on a non-eigenstate, treat it as "FAIL" (something bad happens). Will discuss how to implement measurement as a circuit later later.

Definition 23. An n-qubit stabilizer (Pauli operator) is a $2^n \times 2^n$ matrix of the form

$$P_1 \otimes P_2 \otimes \cdots \otimes P_n, \tag{133}$$

where $k \in \{0, 1, 2, 3\}$ and P_is are one of

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{134}$$

Comment: Easy to see these are Hermitian.