Lecture 23

For this course, if O is measured on a non-eigenstate, treat it as "something more complicated happens". Will discuss how to implement measurement as a circuit later.

Comment: Measure Z on $|0\rangle$. Measure X on $|-\rangle$.

Definition 24. We say two complex matrices A, B of the same dimensions commute if $AB = BA$. We say they anticommute if $AB = -BA$.

Fact 11. A set of n-qubit stabilizers $S_1, \ldots, S_k$ can be *simultaneously measured* if and only if S_i and S_j commute for all $i, j \in [k]$.

Proof. Omitted. This is the physical interpretation of the mathematical fact that mutually commuting diagonalizable matrices can be simultaneously diagonalized. □

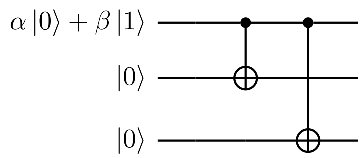
Remark 9. Two stabilizers S_i and S_j either commute or anticommute. This is a consequence of the fact that $XY = -YX$, $XZ = -ZX$, and $YZ = -ZY$ and properties of tensor product. (As you should have seen for yourself in HW3, Q3 — the 4 stabilizers in that homework question can be simultaneously measured.)

Fact 12. Let $|\psi\rangle$ be an n-qubit state, U an n-qubit unitary and P an n-qubit stabilizer. Then if $P|\psi\rangle = |\psi\rangle$ and $PU = \alpha UP$, where $\alpha \in \{-1, +1\}$. Then $PU|\psi\rangle = \alpha U|\psi\rangle$. So measuring P on $U|\psi\rangle$ gives outcome α and the state remains $U|\psi\rangle$.

Proof. Obvious once stated. □

Remark 10. Observe that X, Y, Z are all unitaries. We will instantiate U with X_i, Y_i, Z_i and the identity.

3-qubit bit flip code



The output state is

$$|\psi\rangle := \alpha|000\rangle + \beta|111\rangle. \tag{135}$$

This encoded state protects against all single-qubit bit-flip errors, i.e., X_i.

To see this, consider the following set of two 3-qubit stabilizers that stabilize $|\psi\rangle$:

$$Z_1Z_2 := Z \otimes Z \otimes I \quad \text{and} \quad Z_2Z_3 := I \otimes Z \otimes Z, \tag{136}$$

which can be simultaneously measured since they commute.

Then observe the following error syndrome table

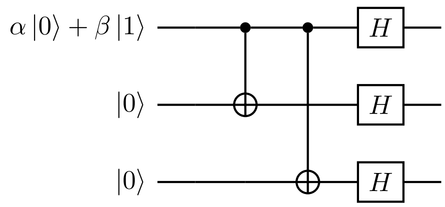
	I	X_1	X_2	X_3
Z_1Z_2	$+$	$-$	$-$	$+$
Z_2Z_3	$+$	$+$	$-$	$-$

Table 1: Error syndrome table for the 3-qubit bit-flip code. The columns correspond to the possible single-qubit bit-flip errors (X_1, X_2, X_3), with I denoting no error. A "+" sign means the stablizer (row label) commutes with the error (column label): by Fact 12 this means when the stabilizer is measured after the error is applies on $|\psi\rangle$ the outcome is $+1$. A "$-$" sign means the stablizer (row label) anticommutes with the error (column label): by Fact 12 this means when the stabilizer is measured after the error is applies on $|\psi\rangle$ the outcome is -1.

Key point: all the columns are distinct, so the error can be deduced from the syndrome.

Unfortunately, this error correcting code does not protect against phase flips, i.e., $U = Z_i$. Because the syndrome will be $(+,+)$ and indistinguishable from no error. This leads to the phase flip code.

3-qubit phase flip code



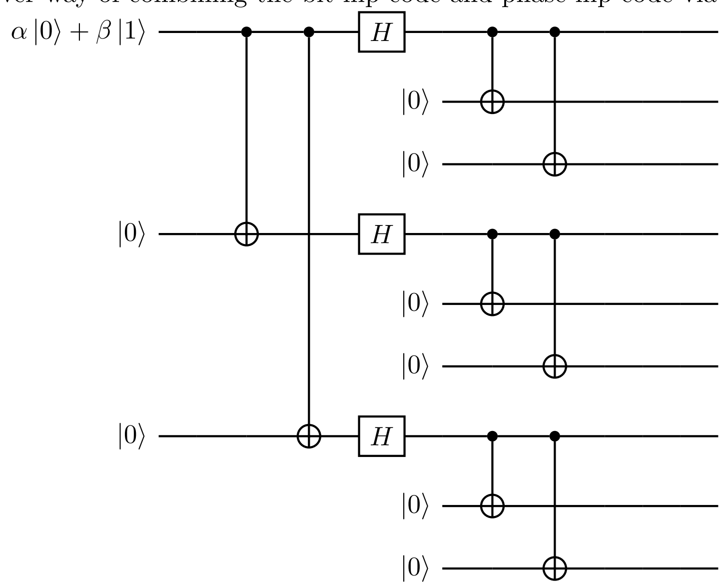
The output state is

$$|\psi\rangle \coloneqq \alpha\,|{+}{+}{+}\rangle + \beta\,|{-}{-}{-}\rangle\,. \tag{137}$$

This now protects against phase flips but not bit flips.

Shor's 9-qubit code

Clever way of combining the bit-flip code and phase-flip code via nesting.



Logical qubits in Shor's nine-qubit code:

$$|0_L\rangle \coloneqq \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) \tag{138}$$

$$|1_L\rangle \coloneqq \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle) \tag{139}$$

The eight (independent)[10] stabilizers of Shor's code:

$$\begin{array}{l} Z_1Z_2 \\ Z_2Z_3 \\ Z_4Z_5 \\ Z_5Z_6 \\ Z_7Z_8 \\ Z_8Z_9 \\ X_1X_2X_3X_4X_5X_6 \\ X_4X_5X_6X_7X_8X_9 \end{array}$$

Proposition 10. *Shor's nine-qubit code corrects any single-qubit X, Z, Y error.*

Proof. In principle can be proven by: draw error syndrome table like before, which has eight rows and $1+3\times 9 = 28$ columns. Check that all columns have distinct signs. (Can think a bit more abstractly than this to simplify the proof.) □

[10] This means you cannot multiply some of them together to get another one and is formally equivalent to the notion of linear independence over $\mathbb{F}_2$ under a certain mapping of n-qubit stabilizers to $\mathbb{F}_2^{2n}$ — don't worry too much about it.