Lecture 5

Comment: Trust accuracy of in-person notes more. Questions from last time: 1. "And the first bit onto the second bit" is an allowed operation. 2. deterministic computation: the subset of randomized computations where the randomized state of the system always stays in a deterministic state.

Can calculate

$$\text{NOT} \cdot \text{CNOT} - \text{CNOT} \cdot \text{NOT} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{pmatrix} \tag{17}$$

Our puzzle boils down to the question: what is the kernel of this matrix?

$$\begin{pmatrix} p \\ p \\ q \\ q \end{pmatrix} \tag{18}$$

But by the normalization constraint: $2p+2q=1$ so $q=1/2-p$. The final constraint non-negativity constraint gives $p, q \geq 0$, i.e., $0 \leq p \leq 1/2$. So the final solution to the puzzle is

$$\begin{pmatrix} p \\ p \\ 1/2-p \\ 1/2-p \end{pmatrix} \tag{19}$$

for $0 \leq p \leq 1/2$.

In the above, we considered deterministic operations on randomized information. Can also consider randomized operations on randomized information. For example, you might want to: with probability p apply CNOT and with probability $1-p$ apply NOT. This corresponds to applying the following matrix on the randomized state

$$p\,\text{CNOT} + (1-p)\,\text{NOT} \tag{20}$$

Definition 2 (Randomized operation aka stochastic operation/matrix)**.** A *randomized operation* acting on n bits is described by a 2^n by 2^n matrix A such that

1. all entries of A are non-negative,
2. the sum of every column is 1

That is, every column of A is a randomized state of n bits.

If *every* column of A is a deterministic state of n bits, then A can also be referred to as a *deterministic operation.*

Comment: COIN FLIP gate, Toffoli gate examples

Proposition 1. *Randomized operations map randomized states to randomized states.*

Proof. Comment: did in class. □

More important concepts: Tensor products of randomized states, tensor products of randomized operations, randomized circuits, measurement, randomized computation, universality. Note tensor product is also referred to as Kronecker product (HW1).

Will not cover these notions in detail here since we will for their analogues in the quantum case. But just want to introduce them and point out they not unique to quantum.

1. \$0. Its randomized state can be written as tensor product of randomized states.
2. NOT_1. Its matrix representation can be written as tensor product of matrices.
3. Randomized circuit = circuit consisting of randomized operations. Examples: $\text{CNOT}_{1,2} \cdot \text{NOT}_1$, $\text{NOT}_{1,2} \cdot \text{CNOT}_{1,2}$, $(p\,\text{CNOT}_{1,2} + (1-p)\,\text{NOT}_1) \cdot \text{NOT}_1$. (Deterministic circuit = circuit consisting of deterministic operations.)
4. Measurement: measuring a randomized state on n bits, $\vec{p}$ is a procedure that returns $x \in \{0,1\}^n$ with probability p_x.
5. Randomized computation = (Input + Ancilla) $\rightarrow$ Randomized circuit $\rightarrow$ Measurement.
6. Universality of the Toffoli gate and COIN FLIP gate for randomized computation. Universality of Toffoli gate for deterministic computation. Universality means: can efficiently perform any computation using any randomized operations (up to low error).