Lecture 7

Quiz: Commutativity? That is, is it true that $A \otimes B = B \otimes A$ for all A, B?

Appendum to last time:

1. The following observation may also be useful for HW1, Q3. Suppose $A \in \mathbb{C}^{d\times d}$, then

$$A\,|i\rangle = \sum_{j=1}^{d} A_{j,i}\,|j\rangle \tag{34}$$

   for all $i \in \{1, \dots, d\}$.

2. A d dimensional quantum state is a length d column vector $\sum_i \alpha_i\,|i\rangle$ such that $\alpha_i \in \mathbb{C}$ and $\sum_{i=1}^{d} |\alpha_i|^2 = 1$. An n-qubit quantum state is 2^n-dimensional.

Example of a unitary matrix

$$\begin{pmatrix} i/2 & -i\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \tag{35}$$

If U is a unitary matrix, then $UU^\dagger = I$ also. (Proof: $U^\dagger U = I$ and U square means U is invertible so right multiplying by U^{-1} gives $U^\dagger = U^{-1}$ then left multiplying by U gives $UU^\dagger = I$.)

Definition 7 (Quantum operation)**.** A *quantum operation* acting on n qubits is described by a 2^n by 2^n unitary matrix.

Common elementary quantum gates:

$$H := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \text{Hadamard gate} \tag{36}$$

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad X \text{ gate} \tag{37}$$

$$Y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Y \text{ gate} \tag{38}$$

$$Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Z \text{ gate} \tag{39}$$

$$T := \begin{pmatrix} 1 & 0 \\ 0 & \exp(i\pi/4) \end{pmatrix} \qquad T \text{ gate} \tag{40}$$

$$\text{CNOT} := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad \text{controlled-NOT gate} \tag{41}$$

$$\text{Toffoli} := \text{the same } 8 \times 8 \text{ matrix as before} \qquad \text{Toffoli gate} \tag{42}$$

A more convenient linear-algebraic definition of CNOT and Toffoli:

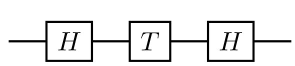
$$\text{CNOT}\colon\ |a\rangle\,|b\rangle = |a\rangle\,|b \oplus a\rangle\,,$$

and

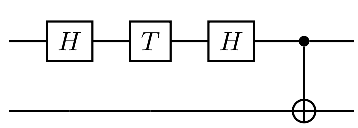
$$\text{Toffoli}\colon\ |a\rangle\,|b\rangle\,|c\rangle = |a\rangle\,|b\rangle\,|c \oplus (a \wedge b)\rangle\,.$$

Definition 8 (Quantum circuit)**.** A quantum circuit is a sequence of quantum operations (typically elementary quantum gates from above).

Consider the following circuits:

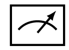


Comment: the above circuit defines a matrix but often only care about how it acts on a fixed input, say, $|0\rangle$.



Definition 9 (Measurement)**.** Measuring an n-qubit quantum state (in the computational basis) $|\psi\rangle = \sum_{x\in\{0,1\}^n} \alpha_x |x\rangle$ refers to a process that returns x with probability $|\alpha_x|^2$.

In a circuit diagram, such measurement often denoted



Then we computed the output distribution of: