Lecture 9

The CHSH game or Bell inequality. This is perhaps the first example of quantum advantage, though not formally in the computational sense because there are *two* parties (Alice and Bob) doing the computation instead of one. Nowadays, it is formalized as quantum advantage in *communication complexity.*[2]

We now follow Watrous notes, Section 4.3. Winning condition

$$a \oplus b = x \wedge y \tag{50}$$

Classical deterministic strategies. Modelling $a\colon \{0,1\} \to \{0,1\}$, $b\colon \{0,1\} \to \{0,1\}$.

x	y	win condition
0	0	$a(0) \oplus b(0) = 0$
0	1	$a(0) \oplus b(1) = 0$
1	0	$a(1) \oplus b(0) = 0$
1	1	$a(1) \oplus b(1) = 1$

Cannot win in all four cases, else XORing the four equations gives $0 = 1$.

Classical randomized strategies Alice and Bob first sample a random variable λ (say real, doesn't matter too much; distribution p_λ) and $a = a_\lambda$ and $= b_\lambda$. Find that maximum winning probability of any randomized strategy with no communication is at most $3/4$. In physics language: any local hidden variable theory can win with probability at most $3/4$. (I'll say more next time: the extra point to make is that we can *guarantee* no communication using the finiteness of the speed of light – which is true if relativity is true.)

Let

$$U_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta. \end{pmatrix} \tag{51}$$

Quantum strategy: use of the following *entangled state* of two qubits, the EPR pair

$$|\phi^+\rangle := \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{52}$$

A two-qubit state is entangled if it cannot be written as a tensor product of two single-qubit states (else it is unentangled). Comment: good exercise to convince yourself that the EPR pair is indeed entangled.

Let A be the operation Alice applies and B the operation Bob applies. Then

1. $x = 0$ means $A = I$, $x = 1$ means $A = H$. Comment: Watrous uses $A = U_{\pi/4}$ when $x = 1$, which is not exactly the same as H, this will change the states just before measurement but won't affect the final winnning probabilities.

2. $y = 0$ means $B = U_{\pi/8}$, $y = 1$ means $B = U_{-\pi/8}$.

Then analyze the case $(x, y) = (0, 0)$ using Figure 4.7 but with Alice *drawn at top*. Then *same as* Watrous notation. Then do $(x, y) = (1, 0)$ using resolution of identity trick:

$$A \otimes B\,|\phi^+\rangle = \langle 00|\, A \otimes B\,|\phi^+\rangle\,|00\rangle + \langle 01|\, A \otimes B\,|\phi^+\rangle\,|01\rangle + \langle 10|\, A \otimes B\,|\phi^+\rangle\,|10\rangle + \langle 11|\, A \otimes B\,|\phi^+\rangle\,|11\rangle \tag{53}$$

Warning. You may be wondering why Watrous draws Alice on the bottom. The reason is given in "Qiskit's qubit ordering convention for circuits" on page 68 of his notes. **We will be following the opposite convention:** qubits at the top of a circuit diagram come left-most in equations (in Watrous' notes, they come right-most). As Watrous mentions, our convention is in fact *more common.* **Please keep this warning in mind whenever you use Watrous' notes as a reference.**

[2]As you'll see later, Alice and Bob can win the game with probability ≈ 0.85 if they use quantum resources, but if they only have classical resources, they would *have to communicate* to win with that probability.